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Superfield approach to (non-)local symmetries for 1-form Abelian gauge theory

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Abstract
We exploit the geometrical superfield formalism to derive the local, covariant and continuous Becchi–Rouet–Stora–Tyutin (BRST) symmetry transformations and the non-local, non-covariant and continuous dual-BRST symmetry transformations for the free Abelian 1-form gauge theory in four (3 + 1)-dimensions (4D) of spacetime. Our discussion is carried out in the framework of BRST invariant Lagrangian density for the above 4D theory in the Feynman gauge. The geometrical origin and interpretation for the (dual-)BRST charges (and the transformations they generate) are provided in the language of translations of some superfields along the Grassmannian directions of the six (4 + 2)-dimensional supermanifold parametrized by the four spacetime and two Grassmannian variables.

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1. Introduction

In the realm of modern developments in theoretical high energy physics, the symmetry transformations (and corresponding generators) have played a very important role. In particular, the local, covariant and continuous *gauge* symmetry transformations have been found to dictate the theoretical description of three (out of four) fundamental interactions of nature. The quantum electrodynamics (QED) is one of the most extensively studied *gauge theories* where the experimental tests and theoretical predictions have matched each other with an unprecedented degree of accuracy in the history of science. One of the most elegant ways of covariantly quantizing such gauge theories (e.g., QED) is the Becchi–Rouet–Stora–Tyutin (BRST) formalism where both the ‘quantum’ gauge (i.e. BRST) invariance and unitarity are respected together at any arbitrary order of perturbation theory. In this formalism, the

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local gauge invariant singular Lagrangian density is extended to include the gauge-fixing and Faddeev–Popov ghost terms. The ensuing Lagrangian density turns out to be endowed with a local, covariant, continuous and nilpotent symmetry transformation which is popularly known as the BRST (or 'quantum' gauge) symmetry transformation [1, 2]. Under this transformation, *the kinetic energy term* corresponding to the gauge field of the Lagrangian density remains invariant (as is the case, even with the usual local gauge symmetry transformation). In the recent past, the (anti-)BRST invariant Lagrangian density for the 1-form (non-)Abelian gauge theories in 4D has been shown to possess a new nilpotent, continuous, non-local and non-covariant BRST type transformations under which *the gauge-fixing term* for the gauge field remains invariant [3–6]. We christen this latter symmetry transformation as the dual(co)-BRST symmetry transformation. This is because of the fact that there exists a deep connection between the kinetic energy term and the gauge-fixing term of the (anti-)BRST invariant Lagrangian density on one hand and the de Rham cohomological operators of the differential geometry on the other hand. For instance, the 2-form $F = \mathrm{d}A$ defines the curvature term $F_{\mu\nu}$ (i.e. $F = \frac{1}{2}(\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu)F_{\mu\nu}$) from which the kinetic energy term is constructed and the 0-form $\delta A = - * \mathrm{d} * A = (\partial \cdot A)$ implies the existence of $(\partial \cdot A)$ which is responsible for the construction of the gauge-fixing term. Here $\delta = - * \mathrm{d}*$ (with $\delta^2 = 0$) and $\mathrm{d} = \mathrm{d}x^\mu\, \partial_\mu$ (with $\mathrm{d}^2 = 0$) are the (co-)exterior derivatives and $*$ is the Hodge duality operation of the differential geometry (see, e.g., [7–11]). Thus, the kinetic energy term and the gauge-fixing term owe their origin to the application of d and δ on the 1-form $A = \mathrm{d}x^\mu\, A_\mu$ in a subtle way. Together with the Laplacian operator $\Delta = \mathrm{d}\delta + \delta\mathrm{d}$, the (co-)exterior derivatives (δ)d form a set $(\mathrm{d}, \delta, \Delta)$ which is popularly known as the set of de Rham cohomological operators. These operators obey an algebra: $\mathrm{d}^2 = 0, \delta^2 = 0, \Delta = (\mathrm{d} + \delta)^2 = \{\mathrm{d}, \delta\}, [\Delta, \mathrm{d}] = 0, [\Delta, \delta] = 0$ showing that Δ is the Casimir operator (see, e.g., [7, 8] for details). The operation of Δ on the 1-form A (i.e. $\Delta A = \mathrm{d}x^\mu\, \Box A_\mu$) leads to the derivation of the equation of motion $\Box A_\mu = 0$ for the gauge-fixed Lagrangian density if we demand the validity of Laplace equation $\Delta A = 0$ for this 1-form gauge theory.

One of the most interesting geometrical approaches to gain an insight into the BRST formalism is the superfield formalism [12–17]. In this approach, the super exterior derivative $\tilde{\mathrm{d}}$ and the Maurer–Cartan equation are exploited together in the so-called horizontality condition[2] where the curvature ($(p + 1)$-form) tensor for the p-form ($p = 1, 2, 3 \ldots$) gauge theory is restricted to be flat along the Grassmannian directions of the $(D + 2)$ dimensional supermanifold that is parametrized by D-number of commuting spacetime variables x^μ and two anti-commuting (i.e. $\theta^2 = 0, \bar{\theta}^2 = 0, \theta\bar{\theta} + \bar{\theta}\theta = 0$) Grassmann variables θ and $\bar{\theta}$. This technique leads to the geometrical interpretation of the conserved and nilpotent $\left(Q^2_{(a)b} = 0\right)$ (anti-)BRST charges $(Q_{(a)b})$ as the translation generators $(\partial/\partial\theta, \partial/\partial\bar{\theta})$ along the Grassmannian directions of the supermanifold. Recently, in a set of papers [19–24], all the super de Rham cohomological operators $(\tilde{\mathrm{d}}, \tilde{\delta}, \tilde{\Delta})$ have been exploited in the generalized versions of the horizontality condition for the 2D free Abelian and self-interacting non-Abelian gauge theories on a four (2+2)-dimensional supermanifold. In this endeavour, the geometrical interpretation for (i) the (anti-)BRST charges and corresponding transformations[3] $s_{(a)b}$, (ii) the (anti-)co-BRST charges and the transformations $s_{(a)d}$ they generate (iii) a bosonic charge (i.e. the anti-commutator of (anti-)BRST and (anti-)co-BRST charges) and corresponding symmetry transformations s_w, (iv) the nilpotency $\left(Q^2_{(a)b} = Q^2_{(a)d} = 0\right)$ of the (anti-)BRST $(Q_{(a)b})$ and (anti-)co-BRST $(Q_{(a)d})$ charges and (v) topological properties of the above 2D

[2] This condition is referred to as the 'soul flatness' condition by Nakanishi and Ojima [18].

[3] We follow here the conventions and notations used by Weinberg [25]. To be precise, in their totality, the nilpotent $(\delta^2_{(A)B} = 0)$ (anti-)BRST transformations $\delta_{(A)B}$ are product of an anti-commuting $(\eta C + C\eta = 0, \eta\bar{C} + \bar{C}\eta = 0)$ spacetime independent parameter η and $s_{(a)b}$ (i.e. $\delta_{(A)B} = \eta s_{(a)b}$) with $s^2_{(a)b} = 0$.

1-form gauge theories, etc, has been provided in the framework of superfield formulation. It is interesting to point out that, for the first time, the Lagrangian density and symmetric energy–momentum tensor for the above *topological field theories* have been shown to correspond to the translation of some composite superfields along the Grassmannian directions of the $(2+2)$-dimensional supermanifold.

As pointed out earlier, the co-BRST symmetry transformations are non-local, non-covariant, continuous and nilpotent [3–6]. Such kind of transformations (and corresponding non-local generators) have not yet been discussed in the purview of the geometrical superfield approach to BRST formalism. The purpose of the present paper is to provide geometrical origin and interpretation for the conserved and nilpotent (co-)BRST charges ($Q_{(d)b}$) (and the transformations they generate) in the framework of superfield formulation applied to the 4D free as well as interacting Abelian (1-form) gauge theory defined on a six $(4+2)$-dimensional supermanifold. In particular, it is a challenging endeavour to provide geometrical origin for the non-local, conserved and nilpotent (anti-)co-BRST charges[4] in the framework of superfield formalism as, to the best of our knowledge, such kind of charges have not yet been discussed in its framework. In the present work, we exploit the super (co-)exterior derivatives $(\tilde{\delta})\tilde{d}$ in the (dual-)horizontality conditions on the $(4\,+\,2)$-dimensional supermanifold and demonstrate that the off-shell nilpotent (anti-)BRST charges (and the nilpotent $\tilde{s}^2_{(a)b} = 0$ transformations they generate) correspond to the translations of some superfields along the $(\theta)\bar{\theta}$ directions of the supermanifold. In the similar fashion, we show that the off-shell nilpotent (anti-)co-BRST charges (and the nilpotent $\tilde{s}^2_{(a)d} = 0$ transformations they generate) *too* correspond to the translation of some superfields along the $(\theta)\bar{\theta}$ directions of the $(4+2)$-dimensional supermanifold. However, there is a clear-cut distinction between them when it comes to the transformations on the fermionic (anti-)ghost fields. Whereas the superfield corresponding to the anti-ghost field $\bar{C}$ becomes chiral under the BRST transformation, it is the superfield corresponding to the ghost field C that turns into chiral under the co-BRST transformation. Just the reverse happens when we consider the derivation of anti-BRST and anti-co-BRST transformations in the framework of superfield formulation. In fact, the superfields corresponding to the (anti-)ghost fields become anti-chiral in the latter case of superfield formulation. In the computation of the Hodge duality $\star$ operation on the six-dimensional supermanifold, we have explained all the steps of our calculation because, to the best of our knowledge, this operation is not quite well known in the literature[5]. We have collected some of the non-trivial results of the $\star$ operation in the appendix also. For the discussion of the geometrical origin of the on-shell nilpotent (anti-)BRST and (anti-)co-BRST transformations, we invoke the (anti-)chiral superfields and establish that the on-shell nilpotent charges correspond to the translation of some variety of the (anti-)chiral superfields along a specific Grassmannian direction of the above supermanifold. In fact, the process of translation of the (anti-)chiral superfields along the Grassmannian directions leads to the derivation of internal on-shell nilpotent symmetries $s_{(a)b}$ and $s_{(a)d}$ on the basic fields of the Lagrangian density of the 4D free Abelian gauge theory. The nilpotency of the on-shell as well as off-shell versions of these charges is captured in the property $(\partial/\partial\theta)^2 = 0$, $(\partial/\partial\bar{\theta})^2 = 0$ of the translation generators $(\partial/\partial\theta)$ and $(\partial/\partial\bar{\theta})$ along the Grassmannian directions of the supermanifold.

The outline of our present paper is as follows. In section 2, we briefly recapitulate the bare essentials of the (anti-)BRST and (anti-)co-BRST symmetries in the Lagrangian formulations to set up the notations and conventions. Section 3 is devoted to the derivation of the off-shell

[4] It will be noted that such (anti-)co-BRST charges exist for the free as well as interacting 4D Abelian 1-form gauge theories and they carry the same expression (cf equation (2.7)) for both these cases.

[5] Private communication with V A Soroka on this topic is gratefully acknowledged.

nilpotent (anti-)BRST and (anti-)co-BRST symmetries in the framework of superfield formulation. The on-shell nilpotent (co-)BRST symmetries are derived by invoking the chiral superfields in section 4. Section 5 deals with the derivation of the on-shell nilpotent anti-BRST and anti-co-BRST symmetries by exploiting the anti-chiral superfields. In section 6, the on-shell nilpotent (anti-)BRST and (anti-)co-BRST symmetries are deduced together by utilizing the general superfield expansions. Finally, in section 7, we make some concluding remarks and point out a few future directions that can be pursued later.

2. Preliminary: (co-)BRST symmetries

We discuss here the on-shell as well as off-shell nilpotent (anti-)BRST and (anti-)co-BRST symmetries in the Lagrangian formalism. To this end in mind, we first begin with the following BRST invariant Lagrangian density for the four $(3+1)$-dimensional (4D) *interacting* Abelian gauge theory[6] in the Feynman gauge (see, e.g., [25–28]):

$$\begin{aligned}\mathcal{L}_b &= -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - \tfrac{1}{2}(\partial\cdot A)^2 + \bar\psi(\mathrm{i}\gamma^\mu D_\mu - m)\psi - \mathrm{i}\,\partial_\mu\bar C\partial^\mu C \\ &\equiv \tfrac{1}{2}(\mathbf{E^2}-\mathbf{B^2}) - \tfrac{1}{2}(\partial\cdot A)^2 + \bar\psi(\mathrm{i}\gamma^\mu D_\mu - m)\psi - \mathrm{i}\,\partial_\mu\bar C\partial^\mu C\end{aligned} \tag{2.1}$$

where $D_\mu = \partial_\mu + \mathrm{ie}\,A_\mu$ is the covariant derivative, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the curvature (field strength) tensor constructed from the vector potential A_μ (with the components $F_{0i} = E_i \equiv \mathbf{E}, F_{ij} = \varepsilon_{ijk}B_k, B_i = \frac{1}{2}\varepsilon_{ijk}F_{jk} \equiv \mathbf{B}$), anti-commuting ($C^2 = \bar C^2 = 0, C\bar C + \bar C C = 0, C\psi + \psi C = 0$, etc) (anti-)ghost fields are required in the theory for maintaining the unitarity and 'quantum' gauge invariance together[7] at any arbitrary order of perturbation theory and $(\bar\psi)\psi$ are the Dirac fields with charge e and mass m (see, e.g., [29] for details). As pointed out earlier in the introduction, we have the gauge-fixing term, the vector potential and the curvature term as the 0-form ($\delta A = - * \mathrm{d} * A = (\partial\cdot A)$), 1-form ($A = \mathrm{d}x^\mu A_\mu$) and 2-form ($F = \mathrm{d}A = \frac{1}{2}(\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu)F_{\mu\nu}$) in our present 4D free (1-form) Abelian gauge theory. The gauge-fixing term and the curvature 2-form are constructed by the application of de Rham cohomological operators δ and d on the 1-form $A = \mathrm{d}x^\mu A_\mu$. It is straightforward to check that under the following on-shell ($\Box C = \Box \bar C = 0$) nilpotent $s^2_{(a)b} = 0$ (anti-)BRST transformations (with $s_b s_{ab} + s_{ab} s_b = 0$) (see, e.g., [25–28] for details)

$$\begin{array}{lllll} s_b A_\mu = \partial_\mu C & s_b C = 0 & s_b \bar C = -\mathrm{i}\,(\partial\cdot A) & s_b\psi = -\mathrm{ie}C\psi & s_b\bar\psi = -\mathrm{ie}\bar\psi C \\ s_{ab} A_\mu = \partial_\mu \bar C & s_{ab}\bar C = 0 & s_{ab} C = +\mathrm{i}\,(\partial\cdot A) & s_{ab}\psi = -\mathrm{ie}\bar C\psi & s_{ab}\bar\psi = -\mathrm{ie}\bar\psi\bar C \end{array} \tag{2.2}$$

the kinetic energy term of the Lagrangian density remains invariant. More precisely, the curvature tensor $F_{\mu\nu}$ itself remains unchanged under the above transformations. In other words, the classical electric field **E** and magnetic field **B** are left intact under the above nilpotent (anti-)BRST transformations. In contrast, under the following on-shell ($\Box C = \Box \bar C = 0$) nilpotent $\left(s^2_{(a)d} = 0\right)$ (anti-)co-BRST $s_{(a)d}$ transformations (with $s_d s_{ad} + s_{ad} s_d = 0$) (see, e.g.,

[6] The free 4D Abelian (1-form) gauge theory is the special case of an interacting theory. We adopt here the conventions and notations such that the 4D flat Minkowski manifold is endowed with a metric: $\eta_{\mu\nu} = \mathrm{diag}\,(+1,-1,-1,-1)$ and totally anti-symmetric Levi-Civita tensor satisfies $\varepsilon_{\mu\nu\lambda\xi}\varepsilon^{\mu\nu\lambda\xi} = -4!$, $\varepsilon_{\mu\nu\lambda\xi}\varepsilon^{\mu\nu\lambda\rho} = -3!\delta^\rho_\xi$ etc with the choice $\varepsilon_{0123} = +1$, $\varepsilon_{0ijk} = \varepsilon_{ijk} = -\varepsilon^{0ijk}$. Here the Greek indices correspond to spacetime directions of the 4D manifold and Latin indices stand for the space directions only. The 3-vectors on the manifold are occasionally denoted by the bold faced letters (e.g., $\mathbf{E}, \mathbf{B}, \mathbf{b^{(1)}}, \mathbf{b^{(2)}}$).

[7] The true strength of the BRST formalism and its (anti-)ghost fields turn up in their full glory for the proof of unitarity in the context of interacting non-Abelian gauge theory where there is a gauge invariant interaction between the quark and gluon fields (see, e.g., [29] for details).

[3] for details)

$$
\begin{aligned}
&s_d A_0 = \mathrm{i}\bar{C} \quad s_d A_i = \mathrm{i}\frac{\partial_0\partial_i}{\nabla^2}\bar{C} \qquad s_d\bar{C} = 0 \qquad s_d\psi = \left(\frac{e}{\nabla^2}\partial_0\bar{C}\right)\psi\\
&s_d C = -A_0 + \frac{\partial_0\partial_i}{\nabla^2}A_i + \frac{e}{\nabla^2}\bar{\psi}\gamma_0\psi \equiv \frac{\partial_i E_i + eJ_0}{\nabla^2} \qquad s_d\bar{\psi} = \bar{\psi}\left(\frac{e}{\nabla^2}\partial_0\bar{C}\right)\\
&s_{ad} A_0 = \mathrm{i}C \qquad s_{ad} A_i = \mathrm{i}\frac{\partial_0\partial_i}{\nabla^2}C \qquad s_{ad} C = 0 \qquad s_{ad}\psi = \left(\frac{e}{\nabla^2}\partial_0 C\right)\psi\\
&s_{ad}\bar{C} = A_0 - \frac{\partial_0\partial_i}{\nabla^2}A_i - \frac{e}{\nabla^2}\bar{\psi}\gamma_0\psi \equiv -\frac{\partial_i E_i + eJ_0}{\nabla^2} \qquad s_{ad}\bar{\psi} = \bar{\psi}\left(\frac{e}{\nabla^2}\partial_0 C\right)
\end{aligned}
\tag{2.3}
$$

it is the gauge-fixing term that remains invariant. More precisely, the term $(\partial\cdot A)$ itself remains unchanged under the above transformation. The salient features, at this stage, are (i) the above Lagrangian density remains invariant (modulo a total derivative) under the (anti-)BRST as well as the (anti-)co-BRST transformations. (ii) The (anti-)BRST transformations leave the 2-form $F = \mathrm{d}A$ invariant. (iii) The (anti-)co-BRST transformations keep the 0-form $(\partial\cdot A) = \delta A$ unaltered. (iv) The 2-form $F = \mathrm{d}A$ and the 0-form $\delta A = (\partial\cdot A)$ are, in some sense, 'Hodge dual' to each other because $\delta = -*\mathrm{d}*$ and d are Hodge dual to each other. (v) The magnetic field $\mathbf{B}$ remains invariant (i.e. $s_{(a)b}B_i = s_{(a)d}B_i = 0$) under all the nilpotent (anti-)BRST and (anti-)co-BRST transformations. (vi) It is obvious that a bosonic symmetry s_w can be obtained from the anti-commutator ($s_w = \{s_b, s_d\} = \{s_{ab}, s_{ad}\}$) of the (anti-)BRST $s_{(a)b}$ and (anti-)co-BRST $s_{(a)d}$ symmetries. However, we shall not discuss here about this symmetry as it is not essential for our present work. An elementary discussion on it can be found in [3] (vii) The (anti-)BRST transformations are local, continuous, covariant and nilpotent but the (anti-) co-BRST transformations are non-local, continuous, non-covariant and nilpotent. (viii) The off-shell nilpotent version of the above nilpotent symmetries has not been discussed *together* in [3–6]. We obtain the off-shell nilpotent version of the above symmetries by invoking a couple of 3-vector auxiliary fields $\mathbf{b^{(1)}}$ and $\mathbf{b^{(2)}}$ and a scalar auxiliary field b_3. They play an important role in linearizing the Lagrangian density (2.1) and, in the process, change it to the following form:

$$
\begin{aligned}
\mathcal{L}_B = b_i^{(1)}E_i - \tfrac{1}{2}(\mathbf{b^{(1)}})^2 - b_i^{(2)}B_i + \tfrac{1}{2}(\mathbf{b^{(2)}})^2 + b_3(\partial\cdot A) + \tfrac{1}{2}(b_3)^2\\
+\bar{\psi}(\mathrm{i}\gamma^\mu D_\mu - m)\psi - \mathrm{i}\,\partial_\mu\bar{C}\partial^\mu C.
\end{aligned}
\tag{2.4}
$$

It is straightforward to check that $\mathbf{b^{(1)}} = \mathbf{E}$, $\mathbf{b^{(2)}} = \mathbf{B}$ and $b_3 = -(\partial\cdot A)$. The off-shell nilpotent version of the (anti-)BRST transformations (2.2) is

$$
\begin{aligned}
&\tilde{s}_b A_\mu = \partial_\mu C \qquad \tilde{s}_b C = 0 \qquad \tilde{s}_b\psi = -\mathrm{i}eC\psi \qquad \tilde{s}_b\bar{C} = \mathrm{i}\,b_3 \qquad \tilde{s}_b b_3 = 0\\
&\tilde{s}_b\mathbf{E} = 0 \qquad \tilde{s}_b\mathbf{B} = 0 \qquad \tilde{s}_b\mathbf{b^{(1)}} = 0 \qquad \tilde{s}_b\mathbf{b^{(2)}} = 0 \qquad \tilde{s}_b\bar{\psi} = -\mathrm{i}e\bar{\psi}C\\
&\tilde{s}_{ab} A_\mu = \partial_\mu\bar{C} \qquad \tilde{s}_{ab}\bar{C} = 0 \qquad \tilde{s}_{ab}\psi = -\mathrm{i}e\bar{C}\psi \qquad \tilde{s}_{ab}C = -\mathrm{i}\,b_3 \qquad \tilde{s}_{ab}b_3 = 0\\
&\tilde{s}_{ab}\mathbf{E} = 0 \qquad \tilde{s}_{ab}\mathbf{B} = 0 \qquad \tilde{s}_{ab}\bar{\psi} = -\mathrm{i}e\bar{\psi}\bar{C} \qquad \tilde{s}_{ab}\mathbf{b^{(1)}} = 0 \qquad \tilde{s}_{ab}\mathbf{b^{(2)}} = 0
\end{aligned}
\tag{2.5}
$$

and that of the (anti-)co-BRST symmetry transformations in (2.3) is

$$
\begin{aligned}
&\tilde{s}_d A_0 = \mathrm{i}\,\bar{C} \qquad \tilde{s}_d A_i = \mathrm{i}\,\frac{\partial_0\partial_i}{\nabla^2}\bar{C} \qquad \tilde{s}_d\bar{C} = 0\\
&\tilde{s}_d C = \frac{\partial_i b_i^{(1)} + eJ_0}{\nabla^2} \qquad \tilde{s}_d\psi = \left(\frac{e}{\nabla^2}\partial_0\bar{C}\right)\psi \qquad \tilde{s}_d\bar{\psi} = \bar{\psi}\left(\frac{e}{\nabla^2}\partial_0\bar{C}\right)\\
&\tilde{s}_d\mathbf{b^{(1)}} = 0 \qquad \tilde{s}_d b_3 = 0 \qquad \tilde{s}_d\mathbf{b^{(2)}} = 0 \qquad \tilde{s}_d(\partial\cdot A) = 0 \qquad \tilde{s}_d\mathbf{B} = 0\\
&\tilde{s}_{ad} A_0 = \mathrm{i}\,C \qquad \tilde{s}_{ad} A_i = \mathrm{i}\,\frac{\partial_0\partial_i}{\nabla^2}C \qquad \tilde{s}_{ad} C = 0
\end{aligned}
$$

$$\tilde{s}_{ad}\bar{C} = -\frac{\partial_i b_i^{(1)} + eJ_0}{\nabla^2} \qquad \tilde{s}_{ad}\psi = \left(\frac{e}{\nabla^2}\partial_0 C\right)\psi \qquad \tilde{s}_{ad}\bar{\psi} = \bar{\psi}\left(\frac{e}{\nabla^2}\partial_0 C\right)$$
$$\tilde{s}_{ad}\mathbf{b^{(1)}} = 0 \qquad \tilde{s}_{ad}b_3 = 0 \qquad \tilde{s}_{ad}\mathbf{b^{(2)}} = 0 \qquad \tilde{s}_{ad}(\partial\cdot A) = 0 \qquad \tilde{s}_{ad}\mathbf{B} = 0. \tag{2.6}$$

In the later sections, we shall see that the auxiliary fields, present in the Lagrangian density (2.4) for the derivations of the off-shell nilpotent (anti-)BRST and (anti-)co-BRST versions of symmetry transformations, will play important roles.

The (non-)local, conserved and on-shell nilpotent generators for the above on-shell nilpotent (co-)BRST transformations can be computed from the Noether conserved current. These, for the free as well as interacting 4D Abelian gauge theories, are [3]

$$\begin{aligned} Q_d &= \int \mathrm{d}^3x \left[\frac{\partial_0(\partial\cdot A)}{\nabla^2}\dot{\bar{C}} - (\partial\cdot A)\bar{C}\right] \\ Q_b &= \int \mathrm{d}^3x \left[\partial_0(\partial\cdot A)C - (\partial\cdot A)\dot{C}\right]. \end{aligned} \tag{2.7}$$

From the above expressions, the (non-)local, nilpotent and conserved charges corresponding to anti-co-BRST symmetries and anti-BRST symmetries can be computed by the substitutions: $C \to \pm\mathrm{i}\bar{C}, \bar{C} \to \pm\mathrm{i}C$ which turn out to be the discrete symmetry transformations for the ghost part of the action. In fact, for the generic field $\Psi(x) = (A_\mu, C, \bar{C})(x)$ of the theory, the conserved charges Q_r generate the following generic transformations:

$$s_r\Psi = -\mathrm{i}\,[\Psi, Q_r]_\pm \qquad r = b, ab, d, ad, w, g \tag{2.8}$$

where Q_g stands for the conserved ghost charge which generates an infinitesimal continuous and global scale transformation for the basic fields of the theory as: $s_g A_\mu = 0, s_g C = -\Lambda C, s_g\bar{C} = +\Lambda\bar{C}$ where Λ is a global parameter. The $(+)-$ signs, as the subscripts on the square bracket, imply (anti-)commutators depending on the generic field Ψ being (fermionic)bosonic in nature. Thus, we note that (i) there are four (non-)local, continuous, (non-)covariant and on-shell as well as off-shell nilpotent (i.e. $s_{(a)b}^2 = s_{(a)d}^2 = \tilde{s}_{(a)b}^2 = \tilde{s}_{(a)d}^2 = 0$) symmetries and a couple of continuous, (non-)local and (non-)covariant bosonic symmetries s_w and s_g in the theory and (ii) the generic transformation in (2.8) is also valid for the off-shell nilpotent (anti-)BRST $\tilde{s}_{(a)b}$ transformations, (anti-)co-BRST $\tilde{s}_{(a)d}$ transformations and the corresponding bosonic $\tilde{s}_w = \{\tilde{s}_{(a)b}, \tilde{s}_{(a)d}\}$ transformations as well as the scale symmetry transformations s_g.

3. Off-shell nilpotent symmetries: general superfield approach

To derive the off-shell nilpotent (anti-)BRST and (anti-)co-BRST symmetries *together* in the framework of superfield formulation, we resort to the most general super expansion for the superfields $B_\mu(x,\theta,\bar{\theta})$, $\Phi(x,\theta,\bar{\theta})$ and $\bar{\Phi}(x,\theta,\bar{\theta})$. These superfields are the generalization of the local fields $A_\mu(x)$, $C(x)$ and $\bar{C}(x)$ of the 4D Lagrangian density (2.1) to a six (4 + 2)-dimensional supermanifold which is parametrized by the four bosonic (even) spacetime (x^μ, $\mu = 0, 1, 2, 3$) coordinates and two (odd) Grassmannian ($\theta^2 = \bar{\theta}^2 = 0, \theta\bar{\theta} + \bar{\theta}\theta = 0$) variables. The most general expansion for the above superfields is

$$\begin{aligned} B_\mu(x,\theta,\bar{\theta}) &= A_\mu(x) + \theta\bar{R}_\mu(x) + \bar{\theta}R_\mu(x) + \mathrm{i}\,\theta\bar{\theta}S_\mu(x) \\ \Phi(x,\theta,\bar{\theta}) &= C(x) + \mathrm{i}\,\theta\bar{b}_3(x) + \mathrm{i}\,\bar{\theta}\mathcal{B}(x) + \mathrm{i}\,\theta\bar{\theta}s(x) \\ \bar{\Phi}(x,\theta,\bar{\theta}) &= \bar{C}(x) + \mathrm{i}\,\theta\bar{\mathcal{B}}(x) + \mathrm{i}\,\bar{\theta}b_3(x) + \mathrm{i}\,\theta\bar{\theta}\bar{s}(x) \end{aligned} \tag{3.1}$$

where the number of degree of freedom associated with both the sets of the bosonic fields ($A_\mu, S_\mu, b_3, \bar{b}_3, \mathcal{B}, \bar{\mathcal{B}}$) and the fermionic fields ($R_\mu, \bar{R}_\mu, C, \bar{C}, s, \bar{s}$) are equal. It should be noted that the local matter fields ψ and $\bar{\psi}$ of the Lagrangian density (2.1) have not been generalized to their counterparts in the language of superfields. This is due to the fact that, to the best of our knowledge, it is *not* known how to obtain the BRST-type symmetry transformations on the matter fields in the framework of superfield formulation. We comment on it in the conclusion (cf section 7) part of our present paper. The most general form of the super exterior derivative $\tilde{\mathrm{d}}$ and the 1-form super connection $\tilde{A}$

$$\begin{aligned}\tilde{\mathrm{d}} &= \mathrm{d}Z^M \partial_M \equiv \mathrm{d}x^\mu\, \partial_\mu + \mathrm{d}\bar{\theta}\partial_{\bar{\theta}} + \mathrm{d}\theta\partial_\theta \\ \tilde{A} &= \mathrm{d}Z^M \tilde{A}_M \equiv \mathrm{d}x^\mu\, B_\mu(x,\theta,\bar{\theta}) + \mathrm{d}\bar{\theta}\Phi(x,\theta,\bar{\theta}) + \mathrm{d}\theta\bar{\Phi}(x,\theta,\bar{\theta})\end{aligned} \tag{3.2}$$

(with $Z^M = x^\mu, \theta, \bar{\theta}$) lead to the following super curvature 2-form ($\tilde{F} = \tilde{\mathrm{d}}\tilde{A}$):

$$\begin{aligned}\tilde{\mathrm{d}}\tilde{A} &= \tfrac{1}{2}(\mathrm{d}Z^M \wedge \mathrm{d}Z^N)F_{MN} \\ &\equiv (\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu)(\partial_\mu B_\nu) - (\mathrm{d}\theta \wedge \mathrm{d}\theta)(\partial_\theta\bar{\Phi}) - (\mathrm{d}\theta \wedge \mathrm{d}\bar{\theta})(\partial_\theta\Phi + \partial_{\bar{\theta}}\bar{\Phi}) \\ &\quad - (\mathrm{d}\bar{\theta}\wedge\mathrm{d}\bar{\theta})(\partial_{\bar{\theta}}\Phi) + (\mathrm{d}x^\mu \wedge \mathrm{d}\theta)(\partial_\mu\bar{\Phi} - \partial_\theta B_\mu) + (\mathrm{d}x^\mu \wedge \mathrm{d}\bar{\theta})(\partial_\mu\bar{\Phi} - \partial_{\bar{\theta}} B_\mu).\end{aligned} \tag{3.3}$$

Now we exploit the horizontality condition ($\tilde{\mathrm{d}}\tilde{A} = \mathrm{d}A$) which physically implies that there are no superspace contributions to the physical electric and magnetic fields described by the 2-form $F = \mathrm{d}A = \frac{1}{2}(\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu)F_{\mu\nu}$ in the usual 4D spacetime. In other words, all the components of F_{MN} with Grassmannian variables θ and/or $\bar{\theta}$ will be flat. This results in the following relationships among the auxiliary fields of expansion in (3.1) and the basic local fields of the Lagrangian density (2.1) (see, e.g., [14, 20] for details):

$$\begin{aligned}&\mathcal{B}(x) = \bar{\mathcal{B}}(x) = 0 \qquad s(x) = \bar{s}(x) = 0 \qquad b_3(x) + \bar{b}_3(x) = 0 \\ &R_\mu(x) = \partial_\mu C(x) \qquad \bar{R}_\mu(x) = \partial_\mu\bar{C}(x) \qquad S_\mu(x) = \partial_\mu b_3(x).\end{aligned} \tag{3.4}$$

With the above relationships, the expansion in (3.1) can be re-expressed in terms of the off-shell nilpotent (anti-)BRST transformations of (2.5) as

$$\begin{aligned}B_\mu(x,\theta,\bar{\theta}) &= A_\mu(x) + \theta(\tilde{s}_{ab}A_\mu(x)) + \bar{\theta}(\tilde{s}_b A_\mu(x)) + \theta\bar{\theta}(\tilde{s}_b\tilde{s}_{ab}A_\mu(x)) \\ \Phi(x,\theta,\bar{\theta}) &= C(x) + \theta(\tilde{s}_{ab}C(x)) + \bar{\theta}(\tilde{s}_b C(x)) + \theta\bar{\theta}(\tilde{s}_b\tilde{s}_{ab}C(x)) \\ \bar{\Phi}(x,\theta,\bar{\theta}) &= \bar{C}(x) + \theta(\tilde{s}_{ab}\bar{C}(x)) + \bar{\theta}(\tilde{s}_b\bar{C}(x)) + \theta\bar{\theta}(\tilde{s}_b\tilde{s}_{ab}\bar{C}(x)).\end{aligned} \tag{3.5}$$

The above expansion, in view of the relationships in (2.8) for the generators of internal transformations, unequivocally makes it clear that the local conserved and off-shell nilpotent (anti-)BRST charges *geometrically* correspond to the translation generators $(\partial/\partial\theta)\partial/\partial\bar{\theta}$ along the $(\theta)\bar{\theta}$ directions of the $(4+2)$-dimensional supermanifold (in the limit $\theta, \bar{\theta} \to 0$) and their off-shell nilpotency is captured in the specific property of the translation generators which obey $(\partial/\partial\theta)^2 = 0$, $(\partial/\partial\bar{\theta})^2 = 0$ (cf section 7).

Now we shall dwell on the derivation of the off-shell nilpotent, continuous, non-local and non-covariant (anti-)co-BRST symmetry transformations of (2.6) in the framework of superfield formulation. To this end in mind, first of all we derive the dual version ($\star\tilde{A} = \star(\mathrm{d}Z^M)A_M$) of the super 1-form connection $\tilde{A}$ defined in (3.2). The resulting dual super form ($\star\tilde{A}$) is, of course, a 5-form in the six $(4+2)$-dimensional supermanifold. The explicit expression for this $\star$ operation on $\tilde{A}$ is

$$\begin{aligned}\star\,\tilde{A} &= \frac{1}{3!}\varepsilon^{\mu\nu\lambda\xi}(\mathrm{d}x_\nu \wedge \mathrm{d}x_\lambda \wedge \mathrm{d}x_\xi \wedge \mathrm{d}\theta \wedge \mathrm{d}\bar{\theta})B_\mu(x,\theta,\bar{\theta}) \\ &\quad + \frac{1}{4!}\varepsilon^{\mu\nu\lambda\xi}(\mathrm{d}x_\mu \wedge \mathrm{d}x_\nu \wedge \mathrm{d}x_\lambda \wedge \mathrm{d}x_\xi \wedge \mathrm{d}\bar{\theta})\bar{\Phi}(x,\theta,\bar{\theta}) \\ &\quad + \frac{1}{4!}\varepsilon^{\mu\nu\lambda\xi}(\mathrm{d}x_\mu \wedge \mathrm{d}x_\nu \wedge \mathrm{d}x_\lambda \wedge \mathrm{d}x_\xi \wedge \mathrm{d}\theta)\Phi(x,\theta,\bar{\theta}).\end{aligned} \tag{3.6}$$

In fact, the above 5-form ($\star\tilde{A}$) has been computed for the purpose of the operation of super co-exterior derivative $\tilde{\delta} = - \star \tilde{d} \star$ on the super 1-form $\tilde{A}$ where the $\star$ operation is the Hodge duality operation defined on the (4+2)-dimensional supermanifold. The following $\star$ operation on the super differentials (dZ^M) has been taken into account in the above computation:

$$\begin{aligned}
\star(dx^\mu) &= \frac{1}{3!}\varepsilon^{\mu\nu\lambda\xi}(dx_\nu \wedge dx_\lambda \wedge dx_\xi \wedge d\theta \wedge d\bar{\theta}) \\
\star(d\theta) &= \frac{1}{4!}\varepsilon^{\mu\nu\lambda\xi}(dx_\mu \wedge dx_\nu \wedge dx_\lambda \wedge dx_\xi \wedge d\bar{\theta}) \\
\star(d\bar{\theta}) &= \frac{1}{4!}\varepsilon^{\mu\nu\lambda\xi}(dx_\mu \wedge dx_\nu \wedge dx_\lambda \wedge dx_\xi \wedge d\theta).
\end{aligned} \tag{3.7}$$

It should be noted that (i) in the denominator, the factorials have been taken corresponding to the 4D spacetime Minkowski manifold because the Grassmannian differentials behave in a completely different manner than the spacetime differentials. (ii) Even though, the 2-form differentials $d\theta \wedge d\theta$ and $d\bar{\theta} \wedge d\bar{\theta}$ do exist in terms of the Grassmannian co-ordinates, they have not been taken into account in the $\star$ operation on the 1-form spacetime differential dx^μ. This is because of the fact that θ and $\bar{\theta}$ directions are the independent directions on the supermanifold which constitute the dual directions for the differential (dx^μ) along with the other three spacetime directions. The latter spacetime (dual) directions are taken into account through the 4D Levi-Civita tensor $\varepsilon^{\mu\nu\lambda\xi}$ defined on the 4D Minkowski manifold. (iii) In the limit $(\theta, \bar{\theta}) \to 0$, we get back the ordinary Hodge duality $*$ operation defined on the 4D Minkowski manifold. (iv) We follow the convention of arranging the spacetime differentials to the left and Grassmannian differentials to the right in all the wedge products. Now we apply the super exterior derivative $\tilde{d}$ on (3.6) which yields the following:

$$\begin{aligned}
\tilde{d} \star \tilde{A} = \frac{1}{3!}&\varepsilon^{\mu\nu\lambda\xi}(dx_\rho \wedge dx_\nu \wedge dx_\lambda \wedge dx_\xi \wedge d\theta \wedge d\bar{\theta})(\partial^\rho B_\mu)(x, \theta, \bar{\theta}) \\
&- \frac{1}{4!}\varepsilon^{\mu\nu\lambda\xi}(dx_\mu \wedge dx_\nu \wedge dx_\lambda \wedge dx_\xi \wedge d\theta \wedge d\bar{\theta})(\partial_\theta \bar{\Phi})(x, \theta, \bar{\theta}) \\
&- \frac{1}{4!}\varepsilon^{\mu\nu\lambda\xi}(dx_\mu \wedge dx_\nu \wedge dx_\lambda \wedge dx_\xi \wedge d\theta \wedge d\bar{\theta})(\partial_\theta \Phi)(x, \theta, \bar{\theta}) \\
&- \frac{1}{4!}\varepsilon^{\mu\nu\lambda\xi}(dx_\mu \wedge dx_\nu \wedge dx_\lambda \wedge dx_\xi \wedge d\bar{\theta} \wedge d\bar{\theta})(\partial_{\bar{\theta}} \bar{\Phi})(x, \theta, \bar{\theta}) \\
&- \frac{1}{4!}\varepsilon^{\mu\nu\lambda\xi}(dx_\mu \wedge dx_\nu \wedge dx_\lambda \wedge dx_\xi \wedge d\theta \wedge d\bar{\theta})(\partial_{\bar{\theta}} \Phi)(x, \theta, \bar{\theta}).
\end{aligned} \tag{3.8}$$

A few noteworthy points at this stage are (i) we have dropped all the terms in the above which possess more than four differentials in terms of the spacetime co-ordinates and more than two differentials in the Grassmann variables. (ii) The origin for the existence of the differentials $d\theta \wedge d\bar{\theta}$ in the first term, second term and the last term is entirely different. That is to say, the latter two are similar but completely different from the first term in their origin. Thus, while taking the another $\star$ on (3.8), this difference will appear. In fact, another $\star$ operation (due to $\tilde{\delta}\tilde{A} = - \star \tilde{d} \star \tilde{A}$) on (3.8) leads to the following:

$$\tilde{\delta}\tilde{A} \equiv - \star \tilde{d} \star \tilde{A} = (\partial \cdot B) - s^{\theta\bar{\theta}}(\partial_\theta \bar{\Phi} + \partial_{\bar{\theta}} \Phi) - s^{\theta\theta}(\partial_\theta \Phi) - s^{\bar{\theta}\bar{\theta}}(\partial_{\bar{\theta}} \bar{\Phi}) \tag{3.9}$$

where coefficients s' are symmetric (i.e. $s^{\theta\bar{\theta}} = s^{\bar{\theta}\theta}$ etc) and we have exploited the following:

$$\begin{aligned}
&\star(dx_\rho \wedge dx_\nu \wedge dx_\lambda \wedge dx_\xi \wedge d\theta \wedge d\bar{\theta})(\partial^\rho B_\mu) = \varepsilon_{\rho\nu\lambda\xi}(\partial^\rho B_\mu) \\
&\star(dx_\mu \wedge dx_\nu \wedge dx_\lambda \wedge dx_\xi \wedge d\theta \wedge d\bar{\theta})(\partial_\theta \bar{\Phi}) = \varepsilon_{\mu\nu\lambda\xi} s^{\theta\bar{\theta}}(\partial_\theta \bar{\Phi}) \\
&\star(dx_\mu \wedge dx_\nu \wedge dx_\lambda \wedge dx_\xi \wedge d\theta \wedge d\bar{\theta})(\partial_{\bar{\theta}} \Phi) = \varepsilon_{\mu\nu\lambda\xi} s^{\theta\bar{\theta}}(\partial_{\bar{\theta}} \Phi).
\end{aligned} \tag{3.10}$$

It is evident that the presence of the symmetric s' in the $\star$ operation depends on the origin of the wedge products $d\theta \wedge d\bar{\theta}$. This has been done to account for the property of the duality $\star$ operation which requires the validity of $\star(\star G) = \pm G$ on any generic superfield G (see, e.g., [30] for details on the ordinary $*$ operations). Thus, the existence of s' keeps track of (i) the origin of the Grassmannian differentials and (ii) the kind of differentials one should get after a couple of successive $\star$ operations on any arbitrary differential super forms (that contain the Grassmann differentials). Some of these $\star$ operations have been collected in the appendix. The application of the dual-horizontality condition ($\tilde{\delta}\tilde{A} = \delta A$) leads to the following:

$$\begin{aligned} &b_3(x) = \bar{b}_3(x) = s(x) = \bar{s}(x) = 0 \qquad \mathcal{B}(x) + \bar{\mathcal{B}}(x) = 0 \\ &(\partial \cdot R)(x) = 0 \qquad (\partial \cdot \bar{R})(x) = 0 \qquad (\partial \cdot S)(x) = 0. \end{aligned} \tag{3.11}$$

It is straightforward to check that the following choices for the free theory:

$$\begin{aligned} &R_0 = \mathrm{i}\bar{C} \qquad R_i = \mathrm{i}\frac{\partial_i \partial_0}{\nabla^2}\bar{C} \qquad \bar{R}_0 = \mathrm{i}C \\ &\bar{R}_i = \mathrm{i}\,\frac{\partial_i \partial_0}{\nabla^2}C \qquad \mathcal{B} = -\mathrm{i}\,\frac{\partial_i b_i^{(1)}}{\nabla^2} \qquad \bar{\mathcal{B}} = +\mathrm{i}\frac{\partial_i b_i^{(1)}}{\nabla^2} \end{aligned} \tag{3.12}$$

do satisfy the above conditions emerging from the dual-horizontality conditions. For the interacting theory, the auxiliary fields can be chosen as: $\mathcal{B}^{(I)} = -\mathrm{i}\big(\partial_i b_i^{(1)} + eJ_0\big)\big/\nabla^2$, $\bar{\mathcal{B}}^{(I)} = +\mathrm{i}\big(\partial_i b_i^{(1)} + eJ_0\big)\big/\nabla^2$. The expressions for R_μ and $\bar{R}_\mu$ in (3.12) remain intact for the interacting case. It is clear that one can*not* obtain the transformations on the matter fields ψ and $\bar{\psi}$ in the present form of the superfield formulation. As far as the determination of $S_\mu(S_\mu^{(I)})$ is concerned, we choose judiciously the following expressions for its components in the case of free and interacting theories:

$$\begin{aligned} &S_0 = \frac{\partial_i b_i^{(1)}}{\nabla^2} \qquad S_i = \frac{\partial_\mathrm{i}\,\partial_0}{\nabla^2}\left(\frac{\partial_j b_j^{(1)}}{\nabla^2}\right) \\ &S_0^{(I)} = \frac{\partial_i b_i^{(1)} + eJ_0}{\nabla^2} \qquad S_i^{(I)} = \frac{\partial_\mathrm{i}\,\partial_0}{\nabla^2}\left(\frac{\partial_j b_j^{(1)} + eJ_0}{\nabla^2}\right). \end{aligned} \tag{3.13}$$

It is worth pointing out that the auxiliary field $\mathbf{b^{(2)}}$ has not been taken into account here because this field and its equivalent (the magnetic field $\mathbf{B}$) do not transform under any of the transformations discussed above. In terms of the above quantities and the transformations (2.6), we obtain the following expansions for the superfields in (3.1):

$$\begin{aligned} B_\mu(x,\theta,\bar{\theta}) &= A_\mu(x) + \theta(\tilde{s}_{ad}A_\mu(x)) + \bar{\theta}(\tilde{s}_d A_\mu(x)) + \theta\bar{\theta}(\tilde{s}_d\tilde{s}_{ad}A_\mu(x)) \\ \Phi(x,\theta,\bar{\theta}) &= C(x) + \theta(\tilde{s}_{ad}C(x)) + \bar{\theta}(\tilde{s}_d C(x)) + \theta\bar{\theta}(\tilde{s}_d\tilde{s}_{ad}C(x)) \\ \bar{\Phi}(x,\theta,\bar{\theta}) &= \bar{C}(x) + \theta(\tilde{s}_{ad}\bar{C}(x)) + \bar{\theta}(\tilde{s}_d\bar{C}(x)) + \theta\bar{\theta}(\tilde{s}_d\tilde{s}_{ad}\bar{C}(x)). \end{aligned} \tag{3.14}$$

It is obvious now that the (anti-)co-BRST charges which are the generators of (anti-)co-BRST transformations in (2.8) are the *translation generators* along the Grassmannian directions of the six-dimensional supermanifold. The nilpotency of these charges (i.e. $Q_{(a)d}^2 = 0$) *geometrically* corresponds to a couple of successive ($(\partial/\partial\theta)^2 = (\partial/\partial\bar{\theta})^2 = 0$) translations along the Grassmannian directions of the six-dimensional supermanifold (cf section 7).

4. On-shell nilpotent (co-)BRST symmetries: chiral superfield formalism

To provide the geometrical origin and interpretation for the on-shell nilpotent (co-)-BRST symmetries ($s_{(d)b}$) and corresponding generators ($Q_{(d)b}$), we resort to the chiral superfield formulation on the four (4 + 2)-dimensional supermanifold. To this end in mind, first of all we generalize the generic local field $\Psi(x) = (A_\mu(x), C(x), \bar{C}(x))$ of the Lagrangian density (2.1) to a chiral $(\partial_\theta \tilde{A}_M(x,\theta,\bar{\theta}) = 0)$ supervector superfield $\tilde{A}^{(c)}_M(x,\bar{\theta}) = \big(B^{(c)}_\mu(x,\bar{\theta}), \Phi^{(c)}(x,\bar{\theta}), \bar{\Phi}^{(c)}(x,\bar{\theta})\big)$ with the following super expansions along the Grassmannian $\bar{\theta}$-direction of the supermanifold:

$$\begin{aligned} B^{(c)}_\mu(x,\bar{\theta}) &= A_\mu(x) + \bar{\theta} R_\mu(x) \\ \Phi^{(c)}(x,\bar{\theta}) &= C(x) + \mathrm{i}\bar{\theta}\mathcal{B}(x) \\ \bar{\Phi}^{(c)}(x,\bar{\theta}) &= \bar{C}(x) + \mathrm{i}\bar{\theta} b_3(x). \end{aligned} \tag{4.1}$$

There are a few relevant points which we summon here: (i) it is obvious that, in the limit $\bar{\theta} \to 0$, we get back the generic field $\Psi(x)$ of the Lagrangian density (2.1). (ii) In general, a superfield on the six (4 + 2)-dimensional supermanifold is parametrized by the superspace variables $Z^M = (x^\mu, \theta, \bar{\theta})$ as discussed earlier. However, for our present discussions, we have chosen $Z^M_{(c)} = (x^\mu, \bar{\theta})$ as the chiral limit of the general Z^M. (iii) The specific choices in the expansion of the superfields $\Phi^{(c)}(x,\bar{\theta})$ and $\bar{\Phi}^{(c)}(x,\bar{\theta})$ have been guided by the transformations in (2.5) and (2.6). (iv) The total number of degrees of freedom for the odd fields $(R_\mu, C, \bar{C})$ and even fields $\big(A_\mu, b_3, \mathcal{B} = -\mathrm{i}\,\partial_i b^{(1)}_i / \nabla^2\big)$ match in the above expansion for the sake of consistency with the basic tenets of supersymmetry. (v) The auxiliary fields $R_\mu, b_3, \mathbf{b^{(1)}}$ will be fixed in terms of the basic fields after the application of the (dual-)horizontality conditions. Some of them can also be fixed by resorting to the equations of motion for the Lagrangian density (2.4). (vi) All the component fields, on the r.h.s. of the above expansion, are functions of the spacetime even variable x^μ alone. (vii) The super expansions in (4.1) are the chiral limit $(\theta \to 0)$ of the most general expansions in (3.1). (viii) The auxiliary field $\mathbf{b^{(2)}}$ has not been taken into the expansion because its equivalent (the magnetic field $\mathbf{B}$) does not transform under (anti-)BRST as well as (anti-)co-BRST transformations.

Now we exploit the horizontality condition $(\tilde{F} = (\tilde{\mathrm{d}}\tilde{A})|_{(c)} = \tilde{\mathrm{d}}A = F)$ w.r.t. (super) exterior derivatives $(\tilde{\mathrm{d}})\mathrm{d}$ in the construction of the (super) curvature 2-form. Physically, the requirement of the horizontality condition implies an imposition that the curvature 2-form in the ordinary spacetime manifold remains unchanged and it is restricted not to get any contribution from the *superspace* variables. The explicit expressions for the l.h.s. and r.h.s. in the horizontality condition $(\tilde{\mathrm{d}}\tilde{A})|_{(c)} = \mathrm{d}A$ are

$$\begin{aligned} (\tilde{\mathrm{d}}\tilde{A})|_{(c)} &= (\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu)\big(\partial_\mu B^{(c)}_\nu\big) + (\mathrm{d}x^\mu \wedge \mathrm{d}\bar{\theta})\big(\partial_\mu \Phi^{(c)} - \partial_{\bar{\theta}} B^{(c)}_\mu\big) - (\mathrm{d}\bar{\theta} \wedge \mathrm{d}\bar{\theta})(\partial_{\bar{\theta}}\Phi^{(c)}) \\ \mathrm{d}A &= (\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu)(\partial_\mu A_\nu) \equiv \tfrac{1}{2}(\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu)(\partial_\mu A_\nu - \partial_\nu A_\mu) \end{aligned} \tag{4.2}$$

where the super exterior derivative (defined in terms of the chiral superspace coordinates) and super connection 1-form (defined in terms of the chiral superfields) are

$$\begin{aligned} \tilde{\mathrm{d}}|_{(c)} &= \mathrm{d}Z^M_{(c)}\partial_M \equiv \mathrm{d}x^\mu\,\partial_\mu + \mathrm{d}\bar{\theta}\partial_{\bar{\theta}} \\ \tilde{A}|_{(c)} &= \mathrm{d}Z^M_{(c)}\tilde{A}^{(c)}_M \equiv \mathrm{d}x^\mu\, B^{(c)}_\mu(x,\bar{\theta}) + \mathrm{d}\bar{\theta}\Phi^{(c)}(x,\bar{\theta}). \end{aligned} \tag{4.3}$$

It is straightforward to check that the horizontality restriction $(\tilde{\mathrm{d}}\tilde{A})|_{(c)} = \mathrm{d}A$ leads to the following relationships:

$$\partial_{\bar{\theta}}\Phi^{(c)} = 0 \to \mathcal{B}(x) \equiv -\mathrm{i}\,\frac{\partial_i b^{(1)}_i(x)}{\nabla^2} = 0 \qquad \partial_{\bar{\theta}} B^{(c)}_\mu = \partial_\mu \Phi^{(c)} \to R_\mu(x) = \partial_\mu C(x) \tag{4.4}$$

and the restriction $\partial_\mu B_\nu - \partial_\nu B_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$ implies $\partial_\mu R_\nu - \partial_\nu R_\mu = 0$ which is readily satisfied by $R_\mu = \partial_\mu C$. It is obvious that the condition $(\tilde{\mathrm{d}}\tilde{A})|_{(c)} = \mathrm{d}A$ does not fix the auxiliary field $b_3(x)$ in terms of the basic fields of the Lagrangian density (2.1). However, the equation of motion for the Lagrangian density (2.4) comes to our rescue as: $b_3(x) = -(\partial \cdot A)(x)$. With these substitutions for the auxiliary fields, the super expansion (3.1) becomes:

$$\begin{aligned} B_\mu^{(c)}(x,\bar\theta) &= A_\mu(x) + \bar\theta \partial_\mu C(x) \equiv A_\mu(x) + \bar\theta(s_b A_\mu(x)) \\ \Phi^{(c)}(x,\bar\theta) &= C(x) + \mathrm{i}\bar\theta(\mathcal{B}(x) = 0) \equiv C(x) + \bar\theta(s_b C(x) = 0) \\ \bar\Phi^{(c)}(x,\bar\theta) &= \bar C(x) - \mathrm{i}\bar\theta(\partial\cdot A)(x) \equiv \bar C(x) + \bar\theta(s_b \bar C(x)). \end{aligned} \tag{4.5}$$

In fact, now the on-shell nilpotent BRST symmetry transformations in (2.2) can be concisely written in terms of the above superfields expansions as

$$s_b B_\mu^{(c)}(x,\bar\theta) = \partial_\mu \Phi^{(c)}(x,\bar\theta) \qquad s_b \Phi^{(c)}(x,\bar\theta) = 0 \qquad s_b \bar\Phi^{(c)}(x,\bar\theta) = -\mathrm{i}(\partial\cdot B)^{(c)}(x,\bar\theta). \tag{4.6}$$

One can readily check that the first transformation in the above equation leads to $s_b A_\mu = \partial_\mu C, s_b C = 0$; the second transformation produces $s_b C = 0$ and the third one generates $s_b \bar C = -\mathrm{i}(\partial\cdot A), s_b(\partial \cdot A) = \Box C$ in terms of the basic fields of Lagrangian density (2.1). It is interesting to check, vis-a-vis equation (2.8), that

$$\frac{\partial}{\partial\bar\theta}\tilde A_M^{(c)}(x,\bar\theta) = -\mathrm{i}[\Psi(x), Q_b]_\pm \equiv s_b\Psi(x) \qquad \tilde A_M^{(c)} = (\Phi, \bar\Phi, B_\mu)^{(c)} \qquad \Psi = (C, \bar C, A_\mu) \tag{4.7}$$

where the brackets $[,]_\pm$ stand for the (anti-)commutators when the generic field Ψ and superfield $\tilde A_M^{(c)}$ are (fermionic)bosonic in nature. Thus, conserved and on-shell nilpotent BRST charge Q_b geometrically turns out to be the translation generator $\partial/\partial\bar\theta$ for the superfields $\tilde A_M^{(c)}$ along the $\bar\theta$-direction of the supermanifold. The process of this translation generates the on-shell nilpotent BRST symmetry transformations on Ψ which correspond to (2.2). In addition, the nilpotency of $s_b^2 = 0$ and $Q_b^2 = 0$ is intimately connected with the property of the square of the translational generator (i.e. $(\partial/\partial\bar\theta)^2 = 0$).

We illustrate now the derivation of the on-shell nilpotent dual(co-)BRST symmetry transformations of (2.3) by exploiting the analogue of the horizontality condition[8] w.r.t. (super) co-exterior derivatives $(\tilde\delta)\delta$ by requiring $(\tilde\delta\tilde A)|_{(c)} = \delta A$. It is obvious that the chiral limit (i.e. $\theta \to 0$) of the most general expression for $\tilde\delta\tilde A$ in equation (3.9) yields the following expression for $(\tilde\delta\tilde A)|_{(c)}$:

$$\mathrm{Lim}_{\theta\to 0}(\tilde\delta\tilde A) = (\tilde\delta\tilde A)|_{(c)} \equiv (\partial\cdot B)^{(c)}(x,\bar\theta) - s^{\bar\theta\bar\theta}\partial_{\bar\theta}\bar\Phi^{(c)}(x,\bar\theta). \tag{4.8}$$

Due to the dual-horizontality requirement, the above equation (defined on the supermanifold) is to be equated with $\delta A = - * \mathrm{d} * A \equiv (\partial\cdot A)$ (defined on the ordinary 4D spacetime manifold). This restriction leads to the following relationships:

$$\partial_{\bar\theta}\bar\Phi^{(c)}(x,\bar\theta) = 0 \to b_3(x) = 0 \qquad (\partial\cdot B)^{(c)}(x,\bar\theta) = (\partial\cdot A)(x) \to (\partial\cdot R)(x) = 0. \tag{4.9}$$

The above condition $(\partial\cdot R) = 0$ is satisfied automatically by the choice $R_0 = \mathrm{i}\bar C, R_i = \mathrm{i}(\partial_i\partial_0/\nabla^2)\bar C$. It is obvious that, in the expansion (4.1), the auxiliary field $\mathcal{B} = -\mathrm{i}\big(\partial_i b_i^{(1)}/\nabla^2\big)$ or $\mathcal{B}^{(I)} = -\mathrm{i}\big(\partial_i b_i^{(1)} + eJ_0/\nabla^2\big)$ for the free as well as interacting theory is not fixed in terms of the basic fields of (2.1) by the dual-horizontality condition. However, the equation of motion

[8] We christen this condition as the dual-horizontality condition because $\tilde{\mathrm{d}}(\mathrm{d})$ and $\tilde\delta(\delta)$ are dual ($\tilde\delta = -\star\tilde{\mathrm{d}}\star$, $\delta = -*\mathrm{d}*$) to each other. Here the Hodge duality operations $*$ and $\star$ are defined on the 4D Minkowski manifold and 6D supermanifold, respectively. The restriction $(\tilde\delta\tilde A)|_{(c)} = \delta A$ amounts to setting equal to zero all the Grassmannian parts of the superscalar $(\tilde\delta\tilde A)|_{(c)}$. On the ordinary even dimensional manifold, the operation $\delta A = - * \mathrm{d} * A$ always yields the covariant gauge-fixing term $(\partial\cdot A)$ (i.e. 0-form) for the 1-form ($A = \mathrm{d}x^\mu A_\mu$) Abelian $U(1)$ gauge theory in any arbitrary spacetime dimension.

for the Lagrangian density (2.4) helps us to get $\mathbf{b^{(1)}} = \mathbf{E}$. Thus, the chiral super expansion (3.1), for the *free* theory, becomes

$$
\begin{aligned}
B_0^{(c)}(x,\bar{\theta}) &= A_0(x) + \bar{\theta}(\mathrm{i}\bar{C}(x)) \equiv A_0(x) + \bar{\theta}(s_d A_0(x)) \\
B_i^{(c)}(x,\bar{\theta}) &= A_i(x) + \bar{\theta}\left(\mathrm{i}\frac{\partial_i \partial_0}{\nabla^2}\bar{C}(x)\right) \equiv A_i(x) + \bar{\theta}(s_d A_i(x)) \\
\Phi^{(c)}(x,\bar{\theta}) &= C(x) + \bar{\theta}\left(\frac{\partial_i E_i(x)}{\nabla^2}\right) \equiv C(x) + \bar{\theta}(s_d C(x)) \\
\bar{\Phi}^{(c)}(x,\bar{\theta}) &= \bar{C}(x) + \bar{\theta}(\mathrm{i}b_3(x) = 0) \equiv \bar{C}(x) + \bar{\theta}(s_d \bar{C}(x) = 0).
\end{aligned}
\tag{4.10}
$$

It is now evident that

$$
\begin{aligned}
&\frac{\partial}{\partial\bar{\theta}}\tilde{A}_M^{(c)}(x,\bar{\theta}) = -\mathrm{i}[\Psi(x), Q_d]_\pm \equiv s_d \Psi(x) \\
&\tilde{A}_M^{(c)}(x,\bar{\theta}) = (\Phi, \bar{\Phi}, B_0, B_i)^{(c)}(x,\bar{\theta}) \qquad \Psi(x) = (C, \bar{C}, A_0, A_i)(x)
\end{aligned}
\tag{4.11}
$$

where the brackets have the same meaning as discussed earlier. This equation shows that *geometrically* the on-shell nilpotent co-BRST charge Q_d is the generator of translation $\partial/\partial\bar{\theta}$ for the chiral superfield $\tilde{A}_M^{(c)}$ along the Grassmannian direction $\bar{\theta}$ of the $(4+2)$-dimensional supermanifold. Furthermore, the on-shell nilpotency conditions $s_d^2 = 0$ and $Q_d^2 = 0$ are connected with the property of the square of the translational generator $(\partial/\partial\bar{\theta})^2 = 0$. The process of the translation of $\tilde{A}_M^{(c)}(x,\bar{\theta}) = (B_0, B_i, \Phi, \bar{\Phi})^{(c)}(x,\bar{\theta})$ along the $\bar{\theta}$-direction produces the on-shell nilpotent co-BRST transformation $s_d\Psi$ for the generic local field $\Psi = (A_\mu, C, \bar{C})$. Thus, there exists a mapping, namely, $s_d \leftrightarrow \partial/\partial\bar{\theta}$.

There is a clear-cut distinction, however, between Q_b and Q_d as far as the translation of the fermionic superfields (or transformations on (anti-)ghost fields $(\bar{C})C$) along $\bar{\theta}$-direction is concerned. For instance, the translation generated by Q_b along $\bar{\theta}$-direction results in the transformation for the anti-ghost field $\bar{C}$ but analogous translation by Q_d leads to the transformation for the ghost field C. In more sophisticated language, the horizontality condition entails upon the chiral superfield $\bar{\Phi}$ to remain chiral but the chiral superfield Φ becomes a local spacetime field (i.e. $\Phi(x,\bar{\theta}) = C(x)$). In contrast, the dual-horizontality condition entails upon the chiral superfield Φ to retain its chirality but the chiral superfield $\bar{\Phi}$ becomes an ordinary local field (i.e. $\bar{\Phi}(x,\bar{\theta}) = \bar{C}(x)$).

5. Anti-BRST and anti-co-BRST symmetries: anti-chiral superfields

To derive the on-shell nilpotent anti-BRST and anti-co-BRST symmetry transformations of (2.2) and (2.3), we resort to the anti-chiral superfields $\tilde{A}_M^{(ac)}(x,\theta) = \big(B_\mu^{(ac)}, \Phi^{(ac)}, \bar{\Phi}^{(ac)}\big)(x,\theta)$ which have the following super expansions along the θ-direction of the supermanifold:

$$
\begin{aligned}
B_\mu^{(ac)}(x,\theta) &= A_\mu(x) + \theta \bar{R}_\mu(x) \\
\Phi^{(ac)}(x,\bar{\theta}) &= C(x) - \mathrm{i}\theta b_3(x) \\
\bar{\Phi}^{(ac)}(x,\theta) &= \bar{C}(x) + \mathrm{i}\theta \bar{\mathcal{B}}(x).
\end{aligned}
\tag{5.1}
$$

These are, in fact, the anti-chiral limit ($\bar{\theta} \to 0$) of the general super expansion (3.1) on the $(4+2)$-dimensional supermanifold with an exception. The change in sign of the expansion for the superfield $\Phi^{(ac)}(x,\theta)$ has been taken for the algebraic convenience which amounts to the replacement $\bar{b}_3(x) \to -b_3(x)$. This choice has been guided by our knowledge of the most general discussion of nilpotent symmetries in section 2. The super exterior derivative $\tilde{\mathrm{d}}|_{(ac)}$ and super connection 1-form $\tilde{A}|_{(ac)}$, for our present discussion, are

$$\begin{aligned} \tilde{\mathrm{d}}|_{(ac)} &= \mathrm{d}Z^M_{(ac)}\partial_M \equiv \mathrm{d}x^\mu\, \partial_\mu + \mathrm{d}\theta\partial_\theta \\ \tilde{A}|_{(ac)} &= \mathrm{d}Z^M_{(ac)}\tilde{A}_M \equiv \mathrm{d}x^\mu\, B^{(ac)}_\mu(x,\theta) + \mathrm{d}\theta\bar{\Phi}^{(ac)}(x,\theta) \end{aligned} \tag{5.2}$$

which are the anti-chiral limit ($\theta \to 0$, $\mathrm{d}\theta \to 0$) of the corresponding general expressions defined in (3.2). Now the imposition of the horizontality condition $(\tilde{\mathrm{d}}\tilde{A})|_{(ac)} = \mathrm{d}A$ implies the restriction that the curvature 2-form $F = \mathrm{d}A$, defined on the ordinary spacetime manifold, remains unchanged. In other words, the superspace contributions to the curvature 2-form are set equal to zero. The following inputs (i.e. the anti-chiral limit of (3.3)):

$$\begin{aligned} (\tilde{\mathrm{d}}\tilde{A})|_{ac} &= (\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu)\big(\partial_\mu B^{(ac)}_\nu\big) + (\mathrm{d}x^\mu \wedge \mathrm{d}\theta)\big(\partial_\mu\bar{\Phi}^{(ac)} - \partial_\theta B^{(ac)}_\mu\big) - (\mathrm{d}\theta \wedge \mathrm{d}\theta)(\partial_\theta\bar{\Phi}^{(ac)}) \\ \mathrm{d}A &= (\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu)(\partial_\mu A_\nu) \equiv \tfrac{1}{2}(\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu)(\partial_\mu A_\nu - \partial_\nu A_\mu) \end{aligned} \tag{5.3}$$

lead to the following relationships due to $\mathrm{d}A = (\tilde{\mathrm{d}}\tilde{A})|_{ac}$:

$$\partial_\theta\bar{\Phi}^{(ac)}(x,\theta) = 0 \to \bar{\mathcal{B}}(x) = 0 \qquad \partial_\mu\bar{\Phi}^{(ac)}(x,\theta) = \partial_\theta B^{(ac)}_\mu(x,\theta) \to \bar{R}_\mu(x) = \partial_\mu\bar{C}(x) \tag{5.4}$$

and $\partial_\mu B^{(ac)}_\nu - \partial_\nu B^{(ac)}_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$ which implies $\partial_\mu\bar{R}_\nu - \partial_\nu\bar{R}_\mu = 0$. The latter requirement is automatically satisfied by $\bar{R}_\mu = \partial_\mu\bar{C}$. It is clear that the above horizontality restriction does not fix $b_3(x)$ in terms of the basic fields of the Lagrangian density (2.1). However, the equation of motion $b_3 = -(\partial\cdot A)$ for the Lagrangian density (2.4) comes to our help. With these insertions, the super expansion (5.1) becomes

$$\begin{aligned} B^{(ac)}_\mu(x,\theta) &= A_\mu(x) + \theta\partial_\mu\bar{C}(x) \equiv A_\mu(x) + \theta(s_{ab}A_\mu(x)) \\ \Phi^{(ac)}(x,\bar{\theta}) &= C(x) + \mathrm{i}\theta(\partial\cdot A)(x) \equiv C(x) + \theta(s_{ab}C(x)) \\ \bar{\Phi}^{(ac)}(x,\theta) &= \bar{C}(x) + \mathrm{i}\theta(\bar{\mathcal{B}}(x) = 0) \equiv \bar{C}(x) + \theta(s_{ab}\bar{C}(x) = 0). \end{aligned} \tag{5.5}$$

It is now straightforward to check that

$$\begin{aligned} &\frac{\partial}{\partial\theta}\tilde{A}^{(ac)}_M(x,\theta) = -\mathrm{i}[\Psi(x), Q_{ab}]_\pm \equiv s_{ab}\Psi(x) \\ &\tilde{A}^{(ac)}_M = (\Phi, \bar{\Phi}, B_\mu)^{(ac)} \qquad \Psi = (C, \bar{C}, A_\mu) \end{aligned} \tag{5.6}$$

where the above brackets have the same interpretation as discussed earlier. This equation shows that *geometrically* the on-shell nilpotent anti-BRST charge Q_{ab} is the generator of translation $\partial/\partial\theta$ for the anti-chiral superfield $\tilde{A}^{(ac)}_M(x,\theta) = (B_\mu, \Phi, \bar{\Phi})^{(ac)}(x,\theta)$ along the θ-direction of the supermanifold. In fact, this process of translation induces the anti-BRST symmetry transformations (i.e. $s_{ab}\Psi$) for the local fields Ψ that are listed in equation (2.2). Thus, there is a mapping $s_{ab} \leftrightarrow \partial/\partial\theta$ between the above two key operators and the nilpotency of the anti-BRST transformation $s^2_{ab} = 0$ (as well as the corresponding charge ($Q^2_{ab} = 0$)) is encoded in the square of the translation generator $(\partial/\partial\theta)^2 = 0$.

For the derivation of the on-shell nilpotent anti-co-BRST symmetry, we shall resort to the anti-chiral superfield formulation. It is straightforward to check that the anti-chiral limit ($\theta \to 0$) of the most general expression (3.9) leads to the following:

$$\mathrm{Lim}_{\bar{\theta}\to 0}(\tilde{\delta}\tilde{A}) = (\tilde{\delta}\tilde{A})|_{(ac)} \equiv (\partial\cdot B)^{(ac)} - s^{\theta\theta}(\partial_\theta\Phi^{(ac)}). \tag{5.7}$$

The restriction $(\tilde{\delta}\tilde{A})|_{(ac)} = \delta A$ (which physically implies an imposition that the 0-form gauge-fixing term $\delta A = (\partial\cdot A)$, defined on the ordinary spacetime manifold, remains unchanged) leads to the following relationships:

$$(\partial_\theta \Phi^{(ac)})(x,\theta) = 0 \rightarrow b_3(x) = 0 \qquad (\partial \cdot B)^{(ac)} = (\partial \cdot A) \rightarrow (\partial \cdot \bar{R}) = 0. \tag{5.8}$$

The condition $(\partial \cdot \bar{R}) = 0$ is readily satisfied by the choice $\bar{R}_0 = \mathrm{i}C,\ R_i = \mathrm{i}\big(\partial_0\partial_i / \nabla^2\big)C$. The dual-horizontality condition $(\tilde{\delta}\tilde{A})|_{(ac)} = \delta A$ does not fix the field $\bar{\mathcal{B}} = +\mathrm{i}\big(\partial_i b_i^{(1)} / \nabla^2\big)$ or $\bar{\mathcal{B}}^{(I)} = +\mathrm{i}\big(\partial_i b_i^{(1)} + eJ_0\big) / \nabla^2$ in terms of the basic fields of free as well as interacting theories. The equation of motion $\mathbf{b^{(1)}} = \mathbf{E}$ for the Lagrangian density (2.4), however, comes to our rescue. The super expansion for the free 4D Abelian theory becomes

$$\begin{aligned}
B_0^{(ac)}(x,\theta) &= A_0(x) + \theta(\mathrm{i}C(x)) \equiv A_0(x) + \theta(s_{ad}A_0(x)) \\
B_i^{(ac)}(x,\theta) &= A_i(x) + \theta\left(\frac{\mathrm{i}\,\partial_0\partial_i}{\nabla^2}\right)C(x) \equiv A_i(x) + \theta(s_{ad}A_i(x)) \\
\Phi^{(ac)}(x,\theta) &= C(x) + \theta(\mathrm{i}\,b_3(x) = 0) \equiv C(x) + \theta(s_{ad}C(x) = 0) \\
\bar{\Phi}^{(ac)}(x,\theta) &= \bar{C}(x) + \theta\left(-\frac{\partial_i E_i}{\nabla^2}\right)(x) \equiv \bar{C}(x) + \theta(s_{ad}\bar{C}(x)).
\end{aligned} \tag{5.9}$$

The geometrical interpretation for the co-BRST charge Q_{ad} is encoded in

$$\begin{aligned}
&\frac{\partial}{\partial\theta}\tilde{A}_M^{(ac)}(x,\theta) = -\mathrm{i}[\Psi(x), Q_{ad}]_\pm \equiv s_{ad}\Psi(x) \\
&\tilde{A}_M^{(ac)} = (\Phi, \bar{\Phi}, B_0, B_i)^{(ac)} \qquad \Psi = (C, \bar{C}, A_0, A_i)
\end{aligned} \tag{5.10}$$

where the brackets $[\ ,\ ]_\pm$ have the same interpretation as explained earlier. It is obvious to note that Q_{ad} turns out to be the translation generator $(\partial/\partial\theta)$ for the anti-chiral superfields $\tilde{A}_M^{(ac)}(x,\theta) = (B_\mu, \Phi, \bar{\Phi})^{(ac)}(x,\theta)$ along the θ-direction of the supermanifold. The action of the on-shell nilpotent transformation operator s_{ad} on the local fields Ψ and the operation of $(\partial/\partial\theta)$ on the anti-chiral superfields $\tilde{A}_M^{(ac)}(x,\theta)$ are inter-related and there exists a mapping $s_{ad} \leftrightarrow (\partial/\partial\theta)$. The nilpotency $s_{ad}^2 = 0$ is connected to $(\partial/\partial\theta)^2 = 0$. Even though both the charges Q_{ad}, Q_{ab} have the similar kind of mapping with the translation generator, there is a clear distinction between them. Whereas the former generates a transformation for the ghost field C through the translation of the superfield Φ, the latter generates the corresponding transformation on the anti-ghost field $\bar{C}$ through the translation of $\bar{\Phi}$ superfield. The direction of translation for the superfields is common for both of them (i.e. the θ-direction of the supermanifold).

6. On-shell nilpotent symmetries: general superfield formulation

It should be emphasized that the on-shell nilpotent (anti-)BRST and (anti-)co-BRST symmetries can be derived *together* if we merge systematically the (anti-)chiral superfields and have the super expansion for the *free* theory as (see, e.g., [19] for details)

$$\begin{aligned}
B_\mu(x,\theta,\bar{\theta}) &= A_\mu(x) + \theta\bar{R}_\mu(x) + \bar{\theta}R_\mu(x) + \mathrm{i}\theta\bar{\theta}S_\mu(x) \\
\Phi(x,\theta,\bar{\theta}) &= C(x) + \mathrm{i}\theta(\partial \cdot A)(x) + \mathrm{i}\bar{\theta}\left(\frac{-\mathrm{i}\,\partial_i E_i}{\nabla^2}\right)(x) + \mathrm{i}\theta\bar{\theta}s(x) \\
\bar{\Phi}(x,\theta,\bar{\theta}) &= \bar{C}(x) + \mathrm{i}\theta\left(\frac{\mathrm{i}\,\partial_i E_i}{\nabla^2}\right)(x) + \mathrm{i}\bar{\theta}(-(\partial \cdot A))(x) + \mathrm{i}\theta\bar{\theta}\bar{s}(x).
\end{aligned} \tag{6.1}$$

For the interacting theory, one has to replace $\partial_i E_i$ in the above by $(\partial_i E_i + eJ_0)$. In our earlier works [19, 24], similar super expansions with the definitions in (3.2) and $\star$ operation defined in (3.7) and (3.10) (together with those given in the appendix), have been exploited in the horizontality condition $(\tilde{F} = \tilde{\mathrm{d}}\tilde{A} = \mathrm{d}A = F)$ as well as in the dual-horizontality conditions

($\tilde{\delta}\tilde{A} = \delta A$) for the 2D free Abelian and self-interacting non-Abelian gauge theories. For our present free as well as interacting 4D theory, the horizontality condition ($\tilde{d}\tilde{A} = dA$) leads to the derivation of the auxiliary fields in terms of the basic fields of the Lagrangian density (2.1) as follows:

$$R_\mu = \partial_\mu C \qquad \bar{R}_\mu = \partial_\mu \bar{C} \qquad S_\mu = -\partial_\mu(\partial \cdot A) \qquad s = \bar{s} = 0. \tag{6.2}$$

Taking the help of the above expressions, the expansions in (6.1) can be expressed in terms of the on-shell nilpotent (anti-)BRST symmetries (2.2) as

$$\begin{aligned} B_\mu(x, \theta, \bar{\theta}) &= A_\mu(x) + \theta(s_{ab} A_\mu(x)) + \bar{\theta}(s_b A_\mu(x)) + \theta\bar{\theta}(s_b s_{ab} A_\mu(x)) \\ \Phi(x, \theta, \bar{\theta}) &= C(x) + \theta(s_{ab} C(x)) + \bar{\theta}(s_b C(x)) + \theta\bar{\theta}(s_b s_{ab} C(x)) \\ \bar{\Phi}(x, \theta, \bar{\theta}) &= \bar{C}(x) + \theta(s_{ab} \bar{C}(x)) + \bar{\theta}(s_b \bar{C}(x)) + \theta\bar{\theta}(s_b s_{ab} \bar{C}(x)). \end{aligned} \tag{6.3}$$

In a similar fashion, the dual horizontality condition ($\tilde{\delta}\tilde{A} = \delta A$) w.r.t. (super) co-exterior derivatives $(\tilde{\delta})\delta$ leads to the following relationships:

$$(\partial \cdot R) = (\partial \cdot \bar{R}) = (\partial \cdot S) = 0 \qquad s = \bar{s} = 0. \tag{6.4}$$

It is evident now that the above relations have solutions in (3.12) and (3.13) which satisfy all the conditions. Thus, in terms of the on-shell nilpotent (anti-)co-BRST symmetry transformations (2.3), the expansion in (6.1) can be written as

$$\begin{aligned} B_\mu(x, \theta, \bar{\theta}) &= A_\mu(x) + \theta(s_{ad} A_\mu(x)) + \bar{\theta}(s_d A_\mu(x)) + \theta\bar{\theta}(s_d s_{ad} A_\mu(x)) \\ \Phi(x, \theta, \bar{\theta}) &= C(x) + \theta(s_{ad} C(x)) + \bar{\theta}(s_d C(x)) + \theta\bar{\theta}(s_d s_{ad} C(x)) \\ \bar{\Phi}(x, \theta, \bar{\theta}) &= \bar{C}(x) + \theta(s_{ad} \bar{C}(x)) + \bar{\theta}(s_d \bar{C}(x)) + \theta\bar{\theta}(s_d s_{ad} \bar{C}(x)). \end{aligned} \tag{6.5}$$

We would like to lay stress on the fact that it is only for the free (1-form) Abelian gauge theory that (anti-)chiral superfields are merged together systematically to produce the on-shell nilpotent (anti-)BRST and (anti-)co-BRST symmetries together in the framework of the geometrical superfield formulation. The same is not true for the non-Abelian gauge theory in any arbitrary dimension of spacetime. In fact, the on-shell nilpotent anti-BRST and anti-co-BRST symmetries do not exist for the non-Abelian gauge theories. For the derivation of the off-shell nilpotent versions of the (anti-)BRST and (anti-)co-BRST symmetries for the non-Abelian gauge theories, one has to introduce another set of auxiliary fields (see, e.g., [18, 25–28] for the details).

7. Conclusions

In the present investigation, we have derived the off-shell as well as on-shell nilpotent versions of the (anti-)BRST and (anti-)co-BRST symmetry transformations for the free 4D 1-form Abelian gauge theory in the framework of geometrical superfield formalism. For this purpose, we have invoked general superfields as well as (anti-)chiral superfields and their corresponding super expansions. We have also derived a mapping between the translation generators $(\partial/\partial\theta, \partial/\partial\bar{\theta})$ (along the $(\theta, \bar{\theta})$ directions of the six $(4+2)$-dimensional supermanifold) and the internal nilpotent transformations of the on-shell variety $s_{(a)b}, s_{(a)d}$ as well as the off-shell variety $\tilde{s}_{(a)b}, \tilde{s}_{(a)d}$ for the Lagrangian density of the theory, as

$$\begin{aligned} &\frac{\partial}{\partial\bar{\theta}} \leftrightarrow s_{(d)b} && \frac{\partial}{\partial\theta} \leftrightarrow s_{ab} && \frac{\partial}{\partial\theta} \leftrightarrow s_{ad} \\ &\tilde{s}_{(d)b} \leftrightarrow \mathrm{Lim}_{\theta,\bar{\theta}\to 0} \frac{\partial}{\partial\bar{\theta}} && \tilde{s}_{ab} \leftrightarrow \mathrm{Lim}_{\theta,\bar{\theta}\to 0} \frac{\partial}{\partial\theta} && \tilde{s}_{ad} \leftrightarrow \mathrm{Lim}_{\theta,\bar{\theta}\to 0} \frac{\partial}{\partial\theta}. \end{aligned} \tag{7.1}$$

This mapping enables us to provide the *geometrical* interpretation for the conserved and nilpotent (anti-)BRST ($Q_{(a)b}$) and (anti-)co-BRST ($Q_{(a)d}$) charges as the translation generators ($\partial/\partial\theta$, $\partial/\partial\bar{\theta}$) along the Grassmannian directions of the supermanifold. Furthermore, it also provides the *geometrical* origin and interpretation for the nilpotency $\big(Q^2_{(a)b} = 0, Q^2_{(a)d} = 0\big)$ property of these charges as a couple of successive translations (i.e. $(\partial/\partial\theta)^2 = (\partial/\partial\bar{\theta})^2 = 0$) along the Grassmannian directions of the supermanifold. The above statements are valid for the off-shell as well as on-shell versions of the (anti-)BRST ($Q_{(a)b}$) and (anti-)co-BRST ($Q_{(a)d}$) charges and their specific nilpotent properties.

One of the interesting features of our investigation is the observation (and its proof) that the (dual-)horizontality conditions on the (anti-)chiral superfields lead to the derivation of the on-shell nilpotent (anti-)BRST and (anti-)co-BRST symmetries (cf sections 4 and 5) that co-exist together (cf section 6) for the Lagrangian density of a 4D free Abelian gauge theory. We have shown that (anti-)chiral superfields can merge consistently in the case of the free 4D Abelian gauge theories and they lead to the derivation of the on-shell nilpotent (anti-)BRST and (anti-)co-BRST symmetries *together*. The same does not happen in the case of self-interacting 2D non-Abelian gauge theory (see, e.g., [24] for details). As emphasized in the introduction, one of the key differences between the (anti-)BRST and (anti-)co-BRST transformations is the fact that whereas the former transformations are local, covariant, continuous and nilpotent, the latter are non-local, non-covariant, continuous and nilpotent. To capture the non-locality and non-covariance of the latter transformations in the framework of superfield approach, we have chosen the non-local auxiliary fields $\mathcal{B} = -\mathrm{i}\big(\partial_i b_i^{(1)} \big/ \nabla^2\big)$ and $\bar{\mathcal{B}} = +\mathrm{i}\big(\partial_i b_i^{(1)} \big/ \nabla^2\big)$ in the super expansion of the superfields $\Phi(x, \theta, \bar{\theta})$ and $\bar{\Phi}(x, \theta, \bar{\theta})$ for the free 4D Abelian gauge theory. For the case of the interacting theory, one can choose instead: $\mathcal{B}^{(I)} = -\mathrm{i}\big(\partial_i b_i^{(1)} + eJ_0\big) \big/ \nabla^2$ and $\bar{\mathcal{B}}^{(I)} = +\mathrm{i}\big(\partial_i b_i^{(1)} + eJ_0\big) \big/ \nabla^2$. In this context, it is worthwhile to mention an interesting observation in [31] that these non-locality and non-covariance disappear in the momentum phase space if we take into account the key ingredients and inputs from the Wigner's little group. In fact, the choice of the reference frame $k^\mu = (\omega, 0, 0, \omega)$ for the propagating massless ($k^2 = 0$) photon along the z-direction of the 4D manifold with energy ω simplifies all the (anti-)commutators of the theory and the whole discussion on the BRST cohomology becomes local and covariant in this particular reference frame (see, e.g., [31] for details).

In the framework of superfield formalism, the non-locality and non-covariance of the transformations on the gauge field A_μ turns up from the conditions $(\partial \cdot R) = (\partial \cdot \bar{R}) = 0$ which emerge due to the dual-horizontality condition (cf (3.9) and (3.11)). This is not the case for the two $(1 + 1)$-dimensional (2D) (i) free Abelian gauge theory [32–34], (ii) interacting Abelian gauge theory where $U(1)$ gauge field couples with the Dirac fields [35, 36], (iii) self-interacting non-Abelian gauge theory [37, 34], etc, where the local and covariant solutions for the the conditions $(\partial \cdot R) = (\partial \cdot \bar{R}) = 0$ do exist as: $R_\mu = -\varepsilon_{\mu\nu}\partial^\nu \bar{C}$ and $\bar{R}_\mu = -\varepsilon_{\mu\nu}\partial^\nu C$ where $\varepsilon_{\mu\nu}$ is the anti-symmetric Levi-Civita tensor in 2D with $\varepsilon_{01} = +1 = \varepsilon^{10}$. Unlike the precise and unique derivation of the (anti-)BRST symmetry transformations due to the horizontality condition $\tilde{\mathrm{d}}\tilde{A} = \mathrm{d}A$, the dual-horizontality condition $\tilde{\delta}\tilde{A} = \delta A$ does not exactly and uniquely lead to the derivation of all the auxiliary fields $R_\mu(x)$ and $\bar{R}_\mu(x)$ in terms of the (anti-)ghost fields $(\bar{C})C$. In fact, for the 4D theory, one has to make judicious choice for R_0, $\bar{R}_0$, R_i and $\bar{R}_i$ in terms of the anti-commuting (anti-)ghost fields for the validity of the conditions $(\partial \cdot R) = (\partial \cdot \bar{R}) = 0$. In a similar fashion, one has to make judicious and clever guess for the expression for S_μ (cf (3.13)) so that it can (i) satisfy $(\partial \cdot S) = 0$ and (ii) be consistent with expansions in (3.5) and (3.14). It can be checked that our choice in (3.13) fulfils both these criteria. In fact, the non-uniqueness of the solutions for $(\partial \cdot R) = (\partial \cdot \bar{R}) = 0$ in the case of 4D 1-form Abelian gauge theory is very interesting because it is primarily this feature of the superfield formulation which is responsible for the existence of several guises of the dual-BRST symmetries that has been discussed extensively in [4]. These different looking symmetries correspond to different choices of R (and $\bar{R}$) such as $R_0 = \mathrm{i}\nabla^2\bar{C}, \mathrm{i}(\nabla^2/\partial_0)\bar{C}$ and $R_i = \mathrm{i}\,\partial_0\partial_i\bar{C}, \mathrm{i}\,\partial_i\bar{C}$ (and $\bar{R}_0 = \mathrm{i}\nabla^2 C, \mathrm{i}(\nabla^2/\partial_0)C, \bar{R}_i = \mathrm{i}\,\partial_0\partial_i C, \mathrm{i}\,\partial_i C)$ etc under which the gauge-fixing term remains invariant. Thus, in some sense, the superfield formulation with the super co-exterior derivative $\tilde{\delta}$ and the corresponding dual-horizontality condition do provide the reason for the existence of several forms of the (non-local, non-covariant, continuous and nilpotent) dual-BRST symmetries for the 1-form Abelian gauge theory.

It has been a long-standing problem to obtain, in a systematic way, the BRST-type transformations (cf (2.2), (2.3), (2.5), (2.6)) on the matter fields ψ and $\bar{\psi}$ in the framework of superfield approach applied to the BRST formalism. So far, the BRST-type transformations on the gauge fields (see, e.g., [12–17] for the 1-form and 2-form free gauge theories) and the ghost fields have been obtained in the superfield formulation. In fact, this is the present status of this approach because, hitherto, only the (dual-)horizontality conditions $\tilde{\delta}\tilde{A} = \tilde{\delta}A, \tilde{\mathrm{d}}\tilde{A} = \mathrm{d}A$ etc (that involve the (super-)gauge fields and super (co-)exterior derivatives) have been exploited in the derivation of the BRST-type symmetries on the gauge- and the ghost fields. In these restrictions, it is obvious that the matter fields ψ and $\bar{\psi}$ play no significant role at all. This is the central reason that, so far, it has not been possible to obtain the BRST-type symmetries on the matter fields in the superfield approach. However, we strongly feel that, the continuity equation $\delta J = 0 \rightarrow \partial_\mu J^\mu = 0$, which involves the 1-form J (i.e. $J = \mathrm{d}x^\mu J_\mu$, with $J_\mu = \bar{\psi}\gamma_\mu\psi$) and the co-exterior derivative δ (i.e. $\delta = -*\mathrm{d}*$), might play an important role in the derivation of the BRST-type transformations on the matter fields. In this restriction, all one has to do is to have the super expansions for the superfields corresponding to the matter fields ψ and $\bar{\psi}$ analogous to (3.1). The insertions of these superfields in the restriction $(\tilde{\delta}\tilde{J} = \delta J = 0)$, corresponding to the continuity equation $(\partial_\mu J^\mu = 0)$, might lead to the derivation of BRST-type transformations on the matter fields. There is another clue which might help in such a derivation. This is connected with the restriction that the interaction term $J^\mu A_\mu$ should remain unchanged in the process of supersymmetrization. In other words, this amounts to the condition : $\tilde{J}^\mu B_\mu = J^\mu A_\mu$ where $\tilde{J}^\mu$ is the current constructed with the superfields corresponding to the matter fields ψ and $\bar{\psi}$ and B_μ is the superfield defined in (3.1). These issues and ideas are under investigation at the moment and the preliminary results are found to be encouraging for QED in 2D.

It is interesting to point out that local, covariant, continuous and (off-shell as well as on-shell) nilpotent versions of the (anti-)BRST and (anti-)co-BRST symmetries have been obtained for the 4D free Abelian 2-form gauge theory defined on the flat Minkowski manifold [38, 39]. Its quasi-topological nature has been discussed in [39] and it has been shown that this theory provides a tractable field theoretical model for the Hodge theory in 4D [38, 39]. The 'extended' BRST cohomology for this theory has been discussed in [40] where the insights from the Wigner's little group play a very crucial role. It would be interesting endeavour to capture the (anti-)BRST and (anti-)co-BRST symmetries for the above 2-form Abelian gauge theory in the framework of superfield formalism and provide geometrical origin for the nilpotent charges in the theory. Such studies might shed some light on the quasi-topological nature (see, e.g., [39]) of this theory in the framework of superfield formalism and it might provide some clue to attempt the nilpotent symmetries of this kind present in the case of non-Abelian 2-form gauge theories. All such understandings of the 2-form gauge theories will furnish some insights into our main goal of understanding the interacting 2-form gauge theories where there is a gauge invariant coupling between the matter fields and the anti-symmetric (2-form) gauge field. Another very interesting endeavour that can be attempted is based on the local $OSp(4\,|\,2)$ supersymmetry and its connection with the extended BRST transformations in the context of gravitational theories where the geometrical superfield approach could be applied (see, e.g., [15] for details). In fact, the extension of our work to the realm of gravitational theories will complete the generality of the application of super co-exterior derivative and the corresponding dual-horizontality condition. These are some of the issues which are under investigation and our results will be reported elsewhere [41].

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Appendix

In addition to the $\star$ operations in (3.7) and (3.10), we collect here some of the $\star$ operations on the wedge products of the super differentials defined on the six (4 + 2)-dimensional supermanifold. We have followed our convention of putting the spacetime differentials to the left and the Grassmannian differentials to the right in every wedge products. Some of these $\star$ operations are

$$\begin{aligned}
&\star(\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\lambda \wedge \mathrm{d}\theta) = \varepsilon^{\mu\nu\lambda\xi}(\mathrm{d}x_\xi \wedge \mathrm{d}\bar\theta)\\
&\star(\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\lambda \wedge \mathrm{d}\bar\theta) = \varepsilon^{\mu\nu\lambda\xi}(\mathrm{d}x_\xi \wedge \mathrm{d}\theta)\\
&\star(\mathrm{d}x_\mu \wedge \mathrm{d}\bar\theta) = \frac{1}{3!}\varepsilon_{\mu\nu\lambda\xi}(\mathrm{d}x^\nu \wedge \mathrm{d}x^\lambda \wedge \mathrm{d}x^\xi \wedge \mathrm{d}\theta)\\
&\star(\mathrm{d}x_\mu \wedge \mathrm{d}\theta) = \frac{1}{3!}\varepsilon_{\mu\nu\lambda\xi}(\mathrm{d}x^\nu \wedge \mathrm{d}x^\lambda \wedge \mathrm{d}x^\xi \wedge \mathrm{d}\bar\theta)\\
&\star(\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}\theta \wedge \mathrm{d}\theta) = \frac{1}{2!}\varepsilon^{\mu\nu\lambda\xi}(\mathrm{d}x_\lambda \wedge \mathrm{d}x_\xi)s^{\theta\theta}\\
&\star[(\mathrm{d}x_\mu \wedge \mathrm{d}x_\nu)s^{\theta\theta}] = \frac{1}{2!}\varepsilon_{\mu\nu\lambda\xi}(\mathrm{d}x^\lambda \wedge \mathrm{d}x^\xi \wedge \mathrm{d}\theta \wedge \mathrm{d}\theta)\\
&\star(\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}\bar\theta \wedge \mathrm{d}\bar\theta) = \frac{1}{2!}\varepsilon^{\mu\nu\lambda\xi}(\mathrm{d}x_\lambda \wedge \mathrm{d}x_\xi)s^{\bar\theta\bar\theta}\\
&\star[(\mathrm{d}x_\mu \wedge \mathrm{d}x_\nu)s^{\bar\theta\bar\theta}] = \frac{1}{2!}\varepsilon_{\mu\nu\lambda\xi}(\mathrm{d}x^\lambda \wedge \mathrm{d}x^\xi \wedge \mathrm{d}\bar\theta \wedge \mathrm{d}\bar\theta)\\
&\star(\mathrm{d}x^\mu \wedge \mathrm{d}\theta \wedge \mathrm{d}\theta) = \frac{1}{3!}\varepsilon^{\mu\nu\lambda\xi}(\mathrm{d}x_\nu \wedge \mathrm{d}x_\lambda \wedge \mathrm{d}x_\xi)s^{\theta\theta}\\
&\star[(\mathrm{d}x_\mu \wedge \mathrm{d}x_\nu \wedge \mathrm{d}x_\lambda)s^{\theta\theta}] = \varepsilon_{\mu\nu\lambda\xi}(\mathrm{d}x^\xi \wedge \mathrm{d}\theta \wedge \mathrm{d}\theta)\\
&\star(\mathrm{d}x^\mu \wedge \mathrm{d}\bar\theta \wedge \mathrm{d}\bar\theta) = \frac{1}{3!}\varepsilon^{\mu\nu\lambda\xi}(\mathrm{d}x_\nu \wedge \mathrm{d}x_\lambda \wedge \mathrm{d}x_\xi)s^{\bar\theta\bar\theta}\\
&\star[(\mathrm{d}x_\mu \wedge \mathrm{d}x_\nu \wedge \mathrm{d}x_\lambda)s^{\bar\theta\bar\theta}] = \varepsilon_{\mu\nu\lambda\xi}(\mathrm{d}x^\xi \wedge \mathrm{d}\bar\theta \wedge \mathrm{d}\bar\theta)\\
&\star(\mathrm{d}\theta \wedge \mathrm{d}\theta) = \frac{1}{4!}\varepsilon^{\mu\nu\lambda\xi}(\mathrm{d}x_\mu \wedge \mathrm{d}x_\nu \wedge \mathrm{d}x_\lambda \wedge \mathrm{d}x_\xi)s^{\theta\theta}\\
&\star[(\mathrm{d}x_\mu \wedge \mathrm{d}x_\nu \wedge \mathrm{d}x_\lambda \wedge \mathrm{d}x_\xi)s^{\theta\theta}] = \varepsilon_{\mu\nu\lambda\xi}(\mathrm{d}\theta \wedge \mathrm{d}\theta)\\
&\star(\mathrm{d}\bar\theta \wedge \mathrm{d}\bar\theta) = \frac{1}{4!}\varepsilon^{\mu\nu\lambda\xi}(\mathrm{d}x_\mu \wedge \mathrm{d}x_\nu \wedge \mathrm{d}x_\lambda \wedge \mathrm{d}x_\xi)s^{\bar\theta\bar\theta}\\
&\star[(\mathrm{d}x_\mu \wedge \mathrm{d}x_\nu \wedge \mathrm{d}x_\lambda \wedge \mathrm{d}x_\xi)s^{\bar\theta\bar\theta}] = \varepsilon_{\mu\nu\lambda\xi}(\mathrm{d}\bar\theta \wedge \mathrm{d}\bar\theta).
\end{aligned}$$

It should be noted that we have not included some of the $\star$ operations on the super differentials containing $(\mathrm{d}\theta \wedge \mathrm{d}\bar{\theta})$ because, as pointed out in section 3, these can arise in two entirely different ways. While taking the $\star$ of such differentials, one has to be careful about the presence and/or absence of $s^{\theta\bar{\theta}}$ as illustrated in (3.10). We would like to emphasize that we have chosen here some of the super differentials where some non-trivialities are present. However, one can easily exploit the above understanding to take the $\star$ operations on differentials of the form $(\mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu})$ etc where only spacetime differentials are present. These operations would be analogous to what we have already done in equation (3.7).

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